Perceptual and Motor Skills, 1979, 48, 607-615.

SUBJECTIVE HYPOTENUSE ESTIMATION: A TEST OF THE PYTHAGOREAN THEOREM[1]

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Summary.—The Pythagorean theorem constitutes a normative model for the estimation of the length of the hypotenuse of a right-angled figure. Tested with functional measurement methods, this mathematically correct model was inappropriate as a descriptive model. The problem was an orientation effect; hypotenuse estimates for vertically oriented figures were larger than those for horizontally oriented figures composed of the same lengths. Because of this discrepancy, no additive model could describe the judgments.

The purpose of this study was to investigate the psychological process by which the length of the hypotenuse of a right angle is estimated. This judgment should depend upon the lengths of the sides; the mathematically correct rule is the Pythagorean theorem. Although the judgment is doubtless perceptual rather than computational from the subject's perspective, correct responses will follow the mathematical rule as if the subject were performing algebra on the lengths he perceives.

To adapt the Pythagorean theorem into a testable psychological model requires writing its constituent elements as subjective values. Subjective values are considered because a perceived length need not be equal to the corresponding physical value. Equation 1 is a rational algebraic model for subjective hypotenuse estimation:

$$R_{ij} = (S_{H_i}{}^2 + S_{V_j}{}^2)^{1/2} \qquad [1]$$

where R_{ij} is the judged hypotenuse of the figure for which S_{V_j} and S_{H_i} are the subjective lengths of the vertical and horizontal dimensions, respectively. In the language of functional measurement (Anderson, 1970), Equation 1 is an additive model. This can be seen by squaring both sides of the equation; the square of the response is given by the sum of two components. The additive model incorporates the standard substantive assumption that the two components do not interact. This independence means that the subjective length of one side of a right angle will not vary with the length of the other side. If these components do interact, the additive model might be expected to fail.

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[3]We wish to thank N. H. Anderson, M. H. Birnbaum, W. J. Dowling, C. R. Gray, and J. C. Shanteau for helpful comments on the manuscript. We especially wish to thank Lola Lopes for suggesting the hypotenuse orientation hypothesis.

Testing the model is straightforward. If the stimulus figures are constructed from a factorial design with the vertical and horizontal dimensions as the factors, then the squares of the responses should plot as parallel lines. This simple test of the model is available because Equation 1 is an additive model. In addition to this graphic test, analysis of variance provides a statistical test (Weiss & Anderson, 1969). An important advantage of these tests of additivity is that they do not require separate estimates of the component lengths. That is, the question of the adequacy of the Pythagorean theorem as a model of the judgmental integration is approached without requiring assessment of the relation between subjective and objective length.

METHOD

Apparatus and Design

Each stimulus card contained either an L-shaped right angle or a rectangle. The subject's task was to produce a length equal to the length between the ends of the sides for a right angle or of the diagonal for a rectangle. The instructions emphasized estimation of the physical length. The missing hypotenuse was always oriented from upper left to lower right. The hypotenuse was missing in order to force the subject to attend to the component lengths; had the hypotenuse been present, the judgment would simply be a length estimate. The distinction between right angles and rectangles was incorporated into the design because Sleight and Austin (1952) had demonstrated that component lengths are perceived differently for open and closed figures.

The stimuli were a set of 16 right angles (open figures) and another set of 16 rectangles (closed figures). Each figure was drawn in black ink (1-mm thick) on a square white tagboard, 24.5 cm on a side. Each set of figures was constructed according to a 4×4 factorial design so that it comprised all possible pairings of sides of length 3, 10, 17, and 24 cm. Each subject was presented with three replications of both stimulus sets in one session.

The stimulus cards were presented on a table whose edge was 20 cm in front of the seated subject. The cards were presented perpendicular to the table, centered and 56 cm in front of the subject. The top of the card was approximately 10 cm below eye level. The subject produced a length by moving a sliding indicator from right to left along an unmarked meter stick, which was placed 8 cm from the front edge of and parallel to the table. The experimenter read the response to the nearest millimeter and recorded it while the subject returned the indicator to the zero point.

Subjects

The subjects were nine eighth-grade students, selected by the principal of their school. None had had training in geometry. Eighth-grade students were used as subjects because they were the oldest readily available persons who had not been exposed to the Pythagorean theorem. After the rational model was seen to be inconsistent with the data produced by the children, an additional set of subjects was run. These were five adult volunteers who were graduate students. The adults, all of whom knew the Pythagorean theorem, were tested to see whether formal training in geometry increased agreement with the model.

Instructions and five practice judgments were presented at the beginning of the experimental session. Subjects were run individually, and responses were self-paced; the average time to complete the three replications was 32 min. for a child and 30 min. for an adult. The stimulus cards were shuffled before each replication was presented.

RESULTS

The mean squared responses for the children are plotted as a two-way design in Fig. 1. The length of the horizontal dimension is on the horizontal axis, with each pair of curves corresponding to one value of the vertical dimension. The lines in Fig. 1 are not parallel; they converge toward the right. Because the Pythagorean model requires parallelism of the squared responses, the model is rejected by these data. A similar plot was drawn for the adult subjects,

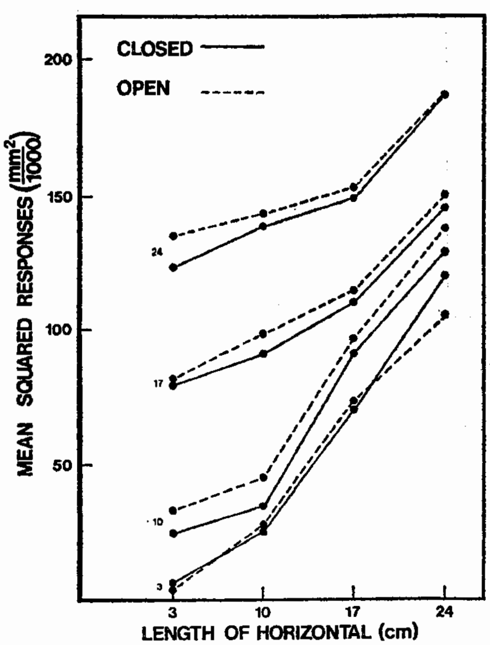

FIG. 1. Mean squared hypotenuse estimates for children's group. Responses were squared, averaged, and then divided by 100 to give the ordinate values. Each pair of curves corresponds to one value of the vertical dimension of the figure. Closed and open figures are rectangles and right angles, respectively.

and it was almost identical in form. Inspection of graphs for individual subjects showed that the violation of the model occurred at the individual level as well.

Statistical Test

The additive model was also disconfirmed via a statistical test. Analysis of variance was applied to the squared responses. The critical test of the model centers on the interaction between the horizontal and vertical dimensions. The F ratios for this stimulus interaction, pooled over open and closed figures, were 14.58 ($df = 9/72$) and 20.34 ($df = 9/36$) for the children and adults, respectively, both significant well beyond the .05 level. Thus, Equation 1 is not a satisfactory description of hypotenuse estimation.

Accuracy of Estimation

Accuracy of the estimates is shown in Fig. 2, which plots the mean response on the vertical axis against the objective value of the hypotenuse on the horizontal axis.[4] Although the children's estimates were essentially linear as a function of the objective values, the discrepancy from correct values consistently increased in magnitude as the stimulus figures increased in size. The adults were more accurate.

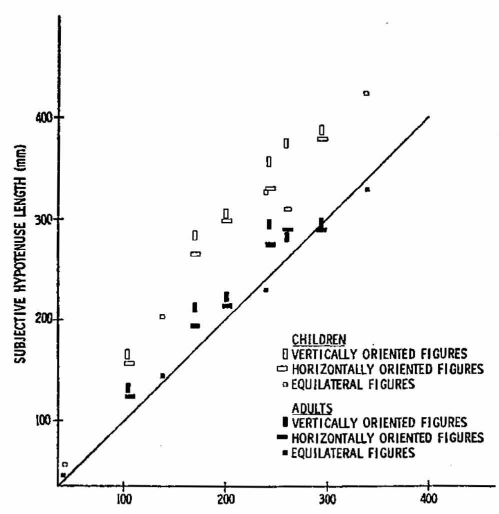

FIG. 2. Mean hypotenuse estimates for children and for adults. Solid line represents objectively correct hypotenuses. Responses to closed and open figures have been pooled.

[4]There is a discrepancy between Figs. 1 and 2, in that generally, squaring a mean response in Fig. 2 (and dividing by 1000) yields a value different from the corresponding value given in Fig. 1. The fact that squaring followed by averaging produces substantially different estimates from averaging followed by squaring indicates that the variance in the responses to a given point is sizeable. However, this variance was primarily inter-individual; it reflects constant differences in estimates, which are irrelevant to the adequacy of the model.

Orientation Effect

It may also be seen in Fig. 2 that vertically longer figures almost always generated larger responses than their normatively equivalent horizontally longer counterparts. The orientation of the symbol on the graph corresponds to the orientation of the stimulus figure.

This orientation effect seems to be different from the classical Horizontal-Vertical (HV) illusion, in which a vertical line appears longer than a horizontal line of equal length. Even if the subjective values for the vertical dimension in Equation 1 were larger than the corresponding values for the horizontal dimension, the present data would still be additive. So long as apparent length depends only upon orientation, the additive character of the model is unaltered; it still generates the parallelism prediction.

Another difference between this illusion and the orientation effect is illustrated by the comparison between responses to the right angles and the rectangles. This distinction between open and closed figures was incorporated into the experimental design as an indirect test of the horizontal-vertical illusion. The illusion has been reported not to operate consistently in closed figures (Sleight & Austin, 1952). If the model succeeded with rectangles but failed with right angles, this would be an indication that the illusion was responsible for the violation of the model. However, Fig. 1 shows that the open-closed figures correspond closely. Statistical analysis of this effect verified its nonsignificance.

Interaction

The present data are consistent with the view that the subjective value of a vertical length increases when it is combined with a shorter horizontal length. Thus, it appears that the independence assumption of the additive model was violated. A direct test of this hypothesis of change in subjective value is afforded by the data of Künnapas (1958). Using right-angled figures as the stimuli, Künnapas asked subjects to estimate the apparent ratio of the shorter to the longer line. His primary conclusion was that the horizontal-vertical illusion appeared clearly in these ratio estimates. Although Künnapas did not evaluate a judgmental model, because he constructed his stimuli from a factorial design, it is possible to apply functional measurement methods to his data to test a ratio or multiplicative model (Anderson & Weiss, 1971). Künnapas' (1958) data have been plotted in a functional measurement format in Fig. 3. The data points generate almost perfectly the diverging fan of lines which is characteristic of multiplicative data.

The illusion can be seen by contrasting the reciprocal of the ratio response for a stimulus pair with the ratio response to the other stimulus pair composed of the same two lengths. As these data are consistent with the normative model for ratio estimates, they furnish support for the independence assumption which is implicit in a multiplicative model. Thus, the Künnapas data support the assumption of independent component lengths, whereas the present data, which employed similar stimuli but a different judgmental task, yielded results that do not. The explanation of this discrepancy may be that the ratio judgment task forces the subject to consider each of the component lengths separately, in order to compare them, while hypotenuse estimation does not. Separate perceptual extraction of the lengths would lead to independence, whereas a global approach to the stimulus allows its components to interact.

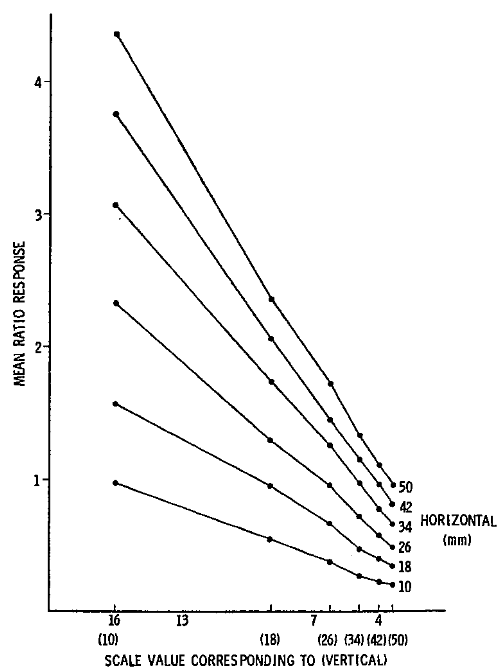

FIG. 3. Mean apparent ratio (Künnapas's data) of the horizontal to the vertical line of L-shaped figures. Values in parentheses on horizontal line are physical lengths of vertical line; each curve corresponds to one value of the length of the vertical line. Abscissa values are spaced in accord with scale values according to Ekman's (1958) method, as given in Table 1 of Künnapas (1958).

Orientation of Hypotenuse

The preceding analyses have focused on the responses as a function of the length and width of the stimulus figures. However, each judgment may also be regarded as an estimate of length of an implicit line, namely, the missing hypotenuse. The implicit line varies in its angle of orientation as a consequence of the disparity between the given sides.

The systematic variation of the length of a line as a function of its orientation has been studied by Pollock and Chapanis (1952). They investigated this aspect of the horizontal-vertical illusion using a reproduction response. The principal finding was that vertical and nearly vertical lines appeared longer than horizontal lines; the maximum proportion of overestimation was for 60° lines.

In order to see whether the effect of orientation of the hypotenuse could account for the nonparallelism in Fig. 1, adjusted hypotenuses were computed for each L-shaped stimulus. The adjustment was that the correct hypotenuse was incremented by an amount determined from the proportion of overestimation for the appropriate inclination as computed from Table 1 of Pollock and Chapanis (1952). The adjusted hypotenuses were consistent with the data in that vertically longer figures generated longer values than their horizontally longer counterparts but were inconsistent in the important respect that while the data converge to the right in Fig. 1, the adjusted hypotenuses diverged in a similar plotting.

Transformation

An alternative possible source of the model's failure is that the response scale was not linearly related to the subject's internal scale. In translating from oblique hypotenuse to horizontally oriented slider, the subject may introduce distortion. Such distortion may be removed by an additivity-inducing transformation (Weiss, 1975).

Accordingly, the responses for each subject were transformed via FUNPOT (Weiss, 1973), a computer program designed to find a polynomial transformation which reduces specified components of a factorial design. In each case, the horizontal-vertical interaction was reduced to non-significance, that is, parallelism was achieved. However, successful transformation does not necessarily argue in favor of the hypothesis of a nonlinear response scale.

Functional measurement assumes that the response scale is at least ordinally correct (Anderson, 1974). Therefore, only a monotone transformation is considered psychologically meaningful; but for 11 of 14 subjects, the additivity-inducing transformation was not monotone. Thus, transforming the response scale is not an appropriate way to resolve the model's failure.

Discussion

No modern-day version of the hecatomb is in order, for the Pythagorean theorem has failed as a model of subjective hypotenuse estimation. The mathematically correct additive model was violated on an individual as well as on a group basis. Use of an additive model leads one to make the strong substantive assumption that the component lengths of a right-angled figure contribute independently to the hypotenuse. Yet the present data suggest that the apparent length of the vertical dimension depends upon the length of the horizontal dimension. This dependence takes the form of an illusion of orientation; vertically longer figures yield larger hypotenuse estimates than horizontally longer figures with the same dimensions. That the interaction is attributable to the orientation effect can be seen by inspection of Fig. 1. If the points for the stimulus figures which were vertically longer, i.e., those on the upper left portion of the graph, were lowered to positions comparable to those of their horizontally longer counterparts, the lines would appear considerably more parallel.

The cause of this orientation effect is unknown. It does not seem to be a consequent of the angle at which a given line is presented, but rather it occurs when one length is perceived in the context of another. A similar peculiarity occurred in a previous study (Anderson & Weiss, 1971) in which subjects estimated the area of rectangles. In that experiment the obvious multiplying model was mildly inconsistent with the data. Although it was not possible to determine precisely the locus of the difficulty, shape effects were a prominent possibility. In accord with the speculation advanced in the discussion of the Künnapas (1958) data, that global judgment of the component lengths leads to interaction, one might expect that judgmental tasks which call for integration of a stimulus figure rather than for analyzing a figure's components will produce violations of the assumption of independence.

The hypotenuse estimates furnish an interesting example of a set of data which should not be rescaled to additivity. The rescaling would have had to be nonmonotone, which is logically doubtful. Also, one would not expect that any transformation should be necessary for length judgments; length yields linear psychophysical scales with almost any response mode (Stevens & Galanter, 1957; Weiss & Anderson, 1969). Indirect evidence of the adequacy of the response scale is given in Fig. 2, in that the responses to the square figures are linearly related to objective hypotenuse length. While transformation is acceptable when the response scale may have distorted the judgments (Weiss & Anderson, 1972), the nonadditivity in the present data is psychologically meaningful. It corresponds to a perceptual effect. While transforming out the interaction might lead to a simple description of the data, a misleadingly simple description of the psychological process would result.

The analytic technique employed here is perhaps worthy of comment. The Pythagorean model given in Equation 1 is an instance of a class of models derived from what Aczél (1966) calls the quasilinear weighted mean. Other models in this class include the geometric mean model, the harmonic mean model, and the root-mean power model, which is a generalization of the Pythagorean model (Weiss, 1975). Any of these models can be tested via analysis of variance after the application of a specified prior transformation. As the commonly applied but seldom tested Euclidean model is a member of this class, the transformation approach may be of considerable utility. One promising avenue for research on multidimensional scaling (e.g., Hyman & Well, 1968) may be to expand the paradigm to incorporate an inverse experiment after the important dimensions have been extracted. In the inverse experiment, a factorial design would be constructed using specific values along the extracted dimensions. The analysis employed here could evaluate whether the stimuli do combine as specified by the postulated Euclidean model.

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Accepted February 13, 1979.